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Stress Fields in General Composite Laminates

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A direct approach is employed to obtain a general boundary integral formulation for the analysis of composite laminates subjected to uniform axial strain. The integral equations governing the problem are directly deduced from the reciprocity theorem, employing the generalized orthotropic elasticity fundamental solutions expressly inferred. The solution is achieved by the boundary element method, which gives, once the traction-free boundary conditions and the interfacial continuity conditions are enforced, a linear system of algebraic equations. The formulation does not present restrictions with regard to the laminate stacking sequence and it does not require any aprioristic assumption. The interlaminar stress field near the free edge of generally stacked composite laminates subjected to uniaxial extension is investigated through this boundary integral equation formulation. The numerical applications show good agreement with those already available in literature, and they demonstrate the accuracy and the efficiency of the proposed method.

Nomenclature

A	= laminate cross section
A_e	= ply cross section
D	= strain operator
D_n	= boundary traction operator
E, Q	= elasticity matrices
E_{ij}	= elasticity stiffness coefficients
f_j	= fundamental solution body forces
l	= laminate length
N	= shape function
p	= nodal tractions
s_1, s_2, s_3	= displacements in the x_1, x_2, x_3 directions, respectively
t	= boundary tractions
$u_j, \varepsilon_j, \sigma_j, t_j$	= fundamental solution displacements, strains, stresses, and tractions, respectively
x_1, x_2, x_3	= coordinate system for the laminate
α_1, α_2	= boundary normal direction cosines
Γ_e	= ply boundary
δ	= nodal displacements
ε	= strain vector
ε_{ij}	= strain components
ε_0	= normal strain in the axial direction, ε_{33}
σ	= stress vector
σ_{ij}	= stress components
σ_{33}	= normal stress in the axial direction

Introduction

THE response of a multilayered, fiber-reinforced composite laminate subjected to static loads has been accurately investigated with the aim of determining the interlaminar stress concentrations near the traction-free edges. Actually, due to the mismatch in elastic properties between plies, a complex three-dimensional stress state showing high gradients in the free edge interlaminar regions arises. These stresses can lead to delamination and failure of the laminate at loads that are lower than the ones at which the structure would fail if only the classical failure mechanisms are involved. The understanding of the free edge effect and the determination of the stress state near the free edge are therefore crucial to correctly describe the laminate behavior and to prevent its early failure.

A great variety of approaches have been used to attempt to calculate the interlaminar stresses due to the free edge effect. The first approach where a complete three-dimensional analysis is performed was presented by Pipes and Pagano,[1] who employed the finite difference technique to obtain the solution of the governing elasticity equations. Many solutions obtained by using the finite element method are available.[2–10] These differ from each other in the formulation, in the kind of employed elements, and in the discretization schemes. In the literature analytical solutions of approximate theories are also present. The techniques employed to achieve these solutions include the perturbation method,[11] series solution,[12,13] Lekhintskii's complex stress potentials coupled with an eigenfunction expansion[14,15] or a polynomial expansion,[16,17] the extension of Reissner's variational principle,[18,19] and the force balance method coupled with the minimum complementary energy principle.[20,21] The stress distributions obtained with these approaches show good agreement between them for points away from the free edge. Considerable disagreement exists instead for points near the free edge location among the various numerical and analytical solutions. This is to be expected as a result of a priori assumptions or because the boundary conditions of the continuum problem have been transformed into conditions on the generalized data. Actually the numerical solution techniques are not able to predict the singular trend in the stress field at the free edge, and hence, to achieve satisfactory solutions a lengthy extrapolation procedure is required. On the other hand, the analytical approximate solutions often rest on assumptions about the problem unknowns that enforce the free edge stress field structure.

In the present paper, the stress field in multilayered composite laminates under uniform axial extension is analyzed on the basis of the integral equation theory.[22,23] The laminate is considered as composed by prismatic elements having different elastic properties. The generic element or ply is treated as homogeneous, and in terms of constitutive equations it is described by a generalized orthotropic law. The integral equations governing the exact elasticity solution of the problem are directly obtained by applying the reciprocity theorem with the fundamental solutions of the generalized orthotropic elasticity.[24–27] The proposed formulation gives a convenient basis for a numerical solution by the boundary element method. Some applications to symmetric cross-ply and angle-ply laminates are presented to check the accuracy and the efficacy of the present method.

Definitions

Consider a composite laminate having cross section A and length l, subjected to uniaxial extension. Let the laminate be composed by generally stacked prismatic plies perfectly bonded at the interface and let it be referred to a coordinate system x_i, $i = 1, 2, 3$, as shown in Fig. 1. The generic ply, having length l and section A_e with boundary Γ_e, is assumed homogeneous, and it obeys a generalized orthotropic law with one of the material symmetry axes

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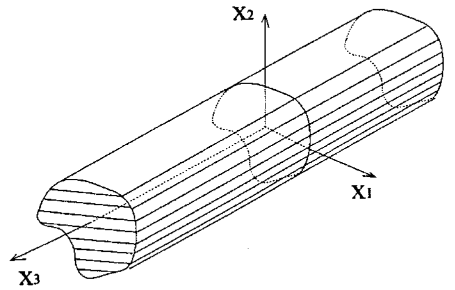

Fig. 1 Coordinate system.

parallel to the x_2 axis. Under the previous assumptions the laminate displacement field can be expressed as

$$s_1 = u_1(x_1, x_2) \tag{1a}$$

$$s_2 = u_2(x_1, x_2) \tag{1b}$$

$$s_3 = u_3(x_1, x_2) + \varepsilon_0 x_3 \tag{1c}$$

where ε_0 is constant all over the laminate section, whereas u_1, u_2, and u_3 depend on x_1 and x_2 only. The strain field associated with the displacements (1) is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \boldsymbol{D u} \tag{2a}$$

$$\varepsilon_{33} = \varepsilon_0 \tag{2b}$$

whereas the stresses are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{31} \\ \sigma_{32} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 & E_{14} & 0 \\ E_{12} & E_{22} & 0 & E_{24} & 0 \\ 0 & 0 & E_{33} & 0 & E_{35} \\ E_{14} & E_{24} & 0 & E_{44} & 0 \\ 0 & 0 & E_{35} & 0 & E_{55} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} E_{16} \\ E_{26} \\ 0 \\ E_{46} \\ 0 \end{bmatrix} \varepsilon_0$$

$$= \boldsymbol{E\varepsilon} + \boldsymbol{Q}\varepsilon_0 \tag{3a}$$

$$\sigma_{33} = [E_{16} \quad E_{26} \quad 0 \quad E_{46} \quad 0]\boldsymbol{\varepsilon} + E_{66}\varepsilon_0 = \boldsymbol{Q}^T\boldsymbol{\varepsilon} + E_{66}\varepsilon_0 \tag{3b}$$

The equilibrium equations, which govern the behavior of each ply, can be expressed as

$$\boldsymbol{D}^T\boldsymbol{\sigma} = \boldsymbol{0} \quad \text{in } A_e \tag{4a}$$

$$\boldsymbol{D}_n\boldsymbol{\sigma} = \boldsymbol{t} \quad \text{on } \Gamma_e \tag{4b}$$

where one has to set

$$\boldsymbol{D}_n = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 \end{bmatrix} \tag{5}$$

In the previous relation α_1 and α_2 are the direction cosines of the outwardly directed normal to the ply section boundary Γ_e, whereas $\boldsymbol{t}$ indicates the surface forces acting on Γ_e.

Integral Equation Formulation

The generic ply, having section A_e with boundary Γ_e, is loaded on its lateral surface by the force system $\boldsymbol{t}$, which is constant along the x_3 axis. Let the elementary prismatic solid be subjected to a fictitious system of body forces $\boldsymbol{f}_j$ depending on x_1 and x_2 only, $\boldsymbol{f}_j = \boldsymbol{f}_j(x_1, x_2)$, and let $\boldsymbol{u}_j$ be a particular system of displacements satisfying the equilibrium equations

$$\boldsymbol{D}^T\boldsymbol{E}\boldsymbol{D}\boldsymbol{u}_j + \boldsymbol{f}_j = \boldsymbol{0} \quad \text{in } A_e \tag{6}$$

Let also $\boldsymbol{\varepsilon}_j$, $\boldsymbol{\sigma}_j$, and $\boldsymbol{t}_j$ be the strain, stress, and traction fields due to $\boldsymbol{u}_j$, respectively. The reciprocity theorem, applied to the actual ply response and to the particular solution described earlier, states

$$\int_{\Gamma_e} \left(\boldsymbol{t}_j^T\boldsymbol{u} - \boldsymbol{u}_j^T\boldsymbol{t}\right) \mathrm{d}\Gamma_e + \int_{A_e} \boldsymbol{f}_j^T\boldsymbol{u} \,\mathrm{d}A_e = \int_{A_e} \left(\boldsymbol{\sigma}_j^T\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_j^T\boldsymbol{\sigma}\right) \mathrm{d}A_e \tag{7}$$

Taking Eqs. (3a) and (3b) into account and bearing in mind that $\varepsilon_{33j} = 0$, from Eq. (7) one obtains

$$\int_{\Gamma_e} \left(\boldsymbol{t}_j^T\boldsymbol{u} - \boldsymbol{u}_j^T\boldsymbol{t}\right) \mathrm{d}\Gamma_e + \int_{A_e} \boldsymbol{f}_j^T\boldsymbol{u} \,\mathrm{d}A_e = -\varepsilon_0 \int_{A_e} \boldsymbol{\varepsilon}_j^T\boldsymbol{Q} \,\mathrm{d}A_e \tag{8}$$

Equation (8) is the fundamental relation for the solution of the problem by the boundary element method. For a point load $\boldsymbol{f}_j$ applied at the point P_0 along the j direction, one obtains the integral equation governing the ply response:

$$\boldsymbol{c}_j^T\boldsymbol{u}(P_0) + \int_{\Gamma_e} \left(\boldsymbol{t}_j^T\boldsymbol{u} - \boldsymbol{u}_j^T\boldsymbol{t}\right) \mathrm{d}\Gamma_e = -\varepsilon_0 \int_{A_e} \boldsymbol{\varepsilon}_j^T\boldsymbol{Q} \,\mathrm{d}A_e \tag{9}$$

The coefficients $\boldsymbol{c}_j$ appearing in Eq. (9) can be obtained according to the following relation:

$$\boldsymbol{c}_j = \int_{A_e} \boldsymbol{f}_j \,\mathrm{d}A_e = -\int_{\Gamma_e} \boldsymbol{t}_j \,\mathrm{d}\Gamma_e \tag{10}$$

The domain integral on the right-hand side of Eq. (9) can also be transformed into an equivalent boundary integral

$$\int_{A_e} \boldsymbol{\varepsilon}_j^T\boldsymbol{Q} \,\mathrm{d}A_e = \int_{\Gamma_e} \boldsymbol{u}_j^T\boldsymbol{q} \,\mathrm{d}\Gamma_e \tag{11}$$

where

$$\boldsymbol{q}^T = [E_{16}\alpha_1 \quad E_{26}\alpha_2 \quad E_{46}\alpha_1] \tag{12}$$

and then Eq. (9) becomes

$$\boldsymbol{c}_j^T\boldsymbol{u}(P_0) + \int_{\Gamma_e} \left(\boldsymbol{t}_j^T\boldsymbol{u} - \boldsymbol{u}_j^T\boldsymbol{t}\right) \mathrm{d}\Gamma_e = -\varepsilon_0 \int_{\Gamma_e} \boldsymbol{u}_j^T\boldsymbol{q} \,\mathrm{d}\Gamma_e \tag{13}$$

Equation (13) is the general integral equation governing the original problem. Once the boundary data are known, it allows one to determine the displacements at each point of the laminate and hence the stress field in a pointwise fashion. For P_0 on the boundary Γ_e Eq. (13) provides for $j = 1, 2, 3$, the relations binding the displacements and the tractions on the boundary of each ply, and it should be sufficient to solve the problem in terms of integral equations. The numerical solution of the formulation is achieved by the boundary element method. The displacements $\boldsymbol{u}$ and the tractions $\boldsymbol{t}$ on the boundary of the generic ply can be expressed in terms of nodal displacements $\boldsymbol{\delta}$ and nodal tractions $\boldsymbol{p}$, respectively, by using properly selected shape functions

$$\boldsymbol{u} = \boldsymbol{N\delta} \tag{14a}$$

$$\boldsymbol{t} = \boldsymbol{Np} \tag{14b}$$

Substituting Eq. (14) into Eq. (13), one has

$$\boldsymbol{c}_j^T\boldsymbol{u}(P_0) + \int_{\Gamma_e} \boldsymbol{t}_j^T\boldsymbol{N} \,\mathrm{d}\Gamma_e\boldsymbol{\delta} - \int_{\Gamma_e} \boldsymbol{u}_j^T\boldsymbol{N} \,\mathrm{d}\Gamma_e\boldsymbol{p} = -\varepsilon_0 \int_{\Gamma_e} \boldsymbol{u}_j^T\boldsymbol{q} \,\mathrm{d}\Gamma_e \tag{15}$$

Equation (15) generally presents six unknowns for each nodal point, whereas only three integral equations can be written for $j = 1, 2, 3$. On the other hand, the nodal points really showing six unknowns are common to the boundaries of the contiguous plies. Collocating Eq. (15) at each nodal point and taking the interfacial continuity conditions into account, one obtains a linear algebraic system that, coupled with the prescribed boundary conditions, provides all of the desired boundary data one needs to characterize the solution.

Fundamental Solutions

To obtain the laminate response by the boundary integral equations, one has to know the generalized orthotropic elasticity fundamental solutions, i.e., the elastic response due to a point load acting at P_0. These fundamental solutions are related to the compliances of the considered ply. According to the theory of anisotropic elasticity,[24] they depend on the roots of the following characteristic equation:

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{16}$$

where

$$a = C_{11}C_{44} - C_{11}^2 \tag{17a}$$

$$b = 2C_{14}(C_{24} + C_{35}) - C_{11}C_{55} - C_{44}(2C_{12} + C_{33}) \tag{17b}$$

$$c = C_{22}C_{44} + C_{55}(2C_{12} + C_{33}) - (C_{24} + C_{35})^2 \tag{17c}$$

$$d = -C_{22}C_{55} \tag{17d}$$

and

$$[C_{rs}] = \boldsymbol{E}^{-1} \tag{18}$$

Assuming that the roots λ_i of Eq. (16) are distinct and positive in sign, as generally happens, the fundamental solutions $\boldsymbol{u}_j$ for a point load in the $j = 1$ and 3 directions are

$$\begin{bmatrix} u_{1j} \\ u_{2j} \\ u_{3j} \end{bmatrix} = \begin{bmatrix} \varphi_1 B_{11} & \varphi_2 B_{12} & \varphi_3 B_{13} \\ \psi_1 B_{21} & \psi_2 B_{22} & \psi_3 B_{23} \\ \varphi_1 B_{31} & \varphi_2 B_{32} & \varphi_3 B_{33} \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} \tag{19}$$

in which

$$B_{1i} = C_{11} - C_{12}/\lambda_i + C_{14}\gamma_i \tag{20a}$$

$$B_{2i} = C_{12} - C_{22}/\lambda_i + C_{24}\gamma_i \tag{20b}$$

$$B_{3i} = C_{14} - C_{24}/\lambda_i + C_{44}\gamma_i \tag{20c}$$

and

$$\gamma_i = \frac{C_{14}\lambda_i - C_{24} - C_{35}}{C_{55} - C_{44}\lambda_i} \tag{21}$$

In Eq. (19) the functions φ_i and ψ_i are defined as

$$\varphi_i(P, P_0) = \frac{-(\ell n\, r_i)}{2\pi} \tag{22a}$$

$$\psi_i(P, P_0) = \frac{-\tan^{-1}(\sqrt{\lambda_i}\, y/x)}{2\pi\sqrt{\lambda_i}} \tag{22b}$$

where

$$x = x_1(P) - x_1(P_0) \tag{23a}$$

$$y = x_2(P) - x_2(P_0) \tag{23b}$$

$$r_i = \sqrt{x^2 + \lambda_i y^2} \tag{23c}$$

The A_{ij} constants are provided by

$$\boldsymbol{A}_j = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ B_{21} & B_{22} & B_{23} \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} c_{1j} \\ 0 \\ c_{3j} \end{bmatrix} \tag{24}$$

The third fundamental solution, associated with the $j = 2$ direction point load, can be again expressed by Eq. (19) if one sets

$$\varphi_i(P, P_0) = -[\tan^{-1}(\sqrt{\lambda_i}\, y/x)]\sqrt{\lambda_i}/2\pi \tag{25a}$$

$$\psi_i(P, P_0) = \frac{\ell n\, r_i}{2\pi} \tag{25b}$$

For this fundamental solution the coefficients A_{i2} need to be calculated from the following relationship:

$$\boldsymbol{A}_2 = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} \begin{bmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} & \lambda_3 B_{13} \\ 1 & 1 & 1 \\ \lambda_1 B_{31} & \lambda_2 B_{32} & \lambda_3 B_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_{22} \\ 0 \end{bmatrix} \tag{26}$$

Generally the fundamental solutions are calculated setting $c_{ij} = \delta_{ij}$ such that when P_0 belongs to smooth boundaries one has $c_{ij} = \delta_{ij}/2$.

Applications

The formulation outlined in the previous sections is applied to some laminate configurations to check its efficiency. Both cross-ply and angle-ply graphite/epoxy laminates are investigated by setting the material elastic properties as follows:

$$\begin{gathered} E_{LL} = 137.9 \text{ GPa}; \qquad E_{TT} = E_{ZZ} = 14.5 \text{ GPa} \\ G_{LT} = G_{LZ} = G_{TZ} = 5.9 \text{ GPa} \\ \nu_{LT} = \nu_{LZ} = \nu_{TZ} = 0.21 \end{gathered} \tag{27}$$

where the subscripts L, T, and Z refer to along fiber, transverse, and thickness directions, respectively. The section geometrical properties are such that the laminate width $2b$ is equal to $16h$, where h is the ply thickness (see Fig. 2). Results from the present method for the distribution of interlaminar stresses of $[0/90]_s$, $[90/0]_s$, and $[45/-45]_s$ laminate symmetric lay-ups, due to uniform axial strain, are presented and compared with the data available in literature. Owing to the symmetry, only a quarter of the laminate section is considered in the analysis, and for each ply the discretization shown in Fig. 3 is used. The influence matrices are calculated by Gauss

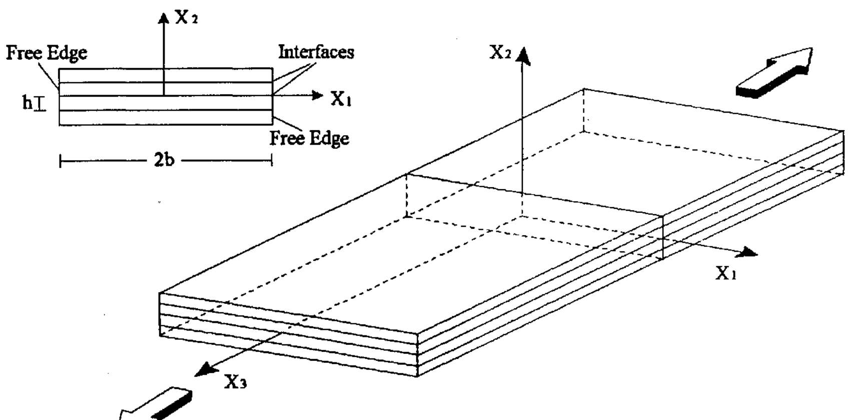

Fig. 2 Laminate configuration.

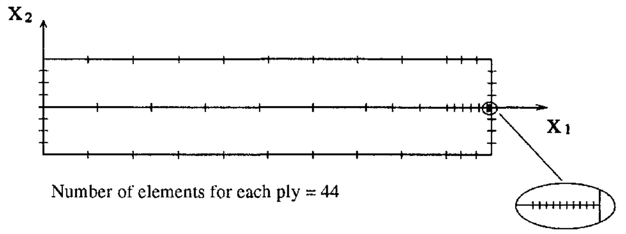

Fig. 3 Boundary element method discretization for a quarter of the laminate section.

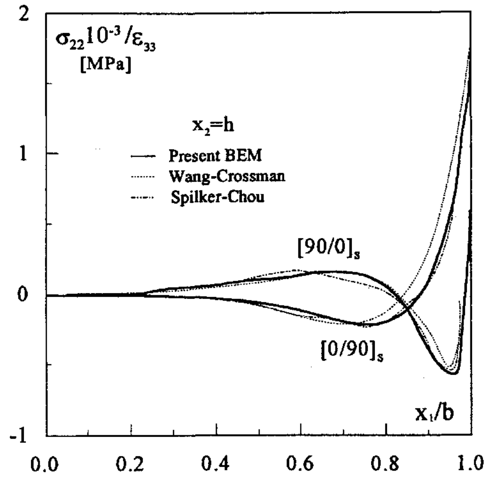

Fig. 4 The σ_{22} distribution for $[0/90]_s$ and $[90/0]_s$ laminates.

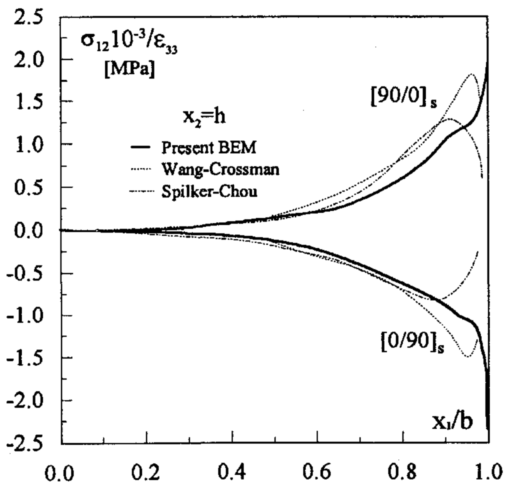

Fig. 5 The σ_{12} distribution for $[0/90]_s$ and $[90/0]_s$ laminates.

quadrature with linear interpolation of the unknown boundary data; after that an adaptive integration scheme has been applied to take the kernel singularities into account.[25–28] At the ply's corners the c_{ij} coefficients are numerically evaluated according to Eq. (10). Note that the assumption of symmetric stress tensor at the free edge requires one to consider a linear combination of the two equations obtained for a point load applied at the interface corner and directed along the x_1 axis to correctly satisfy the free edge boundary conditions. In Figs. 4 and 5 the σ_{22} and σ_{12} distributions with respect to x_1, at $x_2 = h$, for $[0/90]_s$ and $[90/0]_s$ cross-ply laminates are shown, respectively. In Figs. 6 and 7 the σ_{22} and σ_{23} interlaminar stresses across the width at the interface of a $[45/-45]_s$ angle-ply laminate are plotted, respectively. The present solutions compare well with those obtained using different techniques. They also demonstrate how the proposed method allows one to obtain very good accuracy associated with remarkable computational advantages. These are due to the features of the formulation, which requires a boundary-

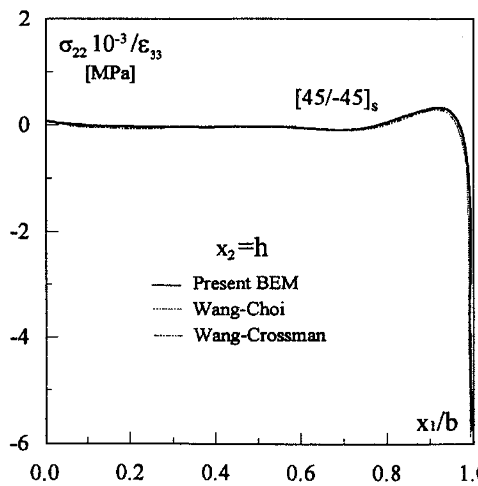

Fig. 6 The σ_{22} distribution for the $[45/-45]_s$ laminate.

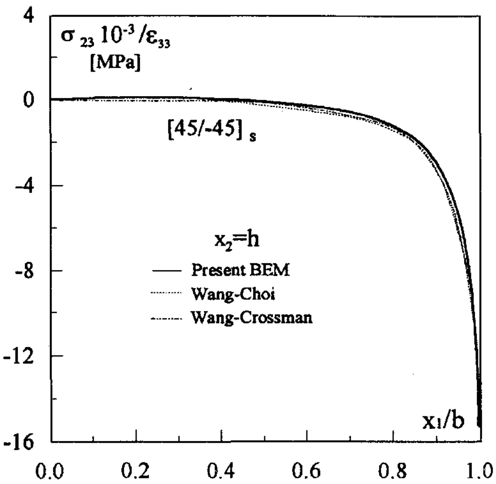

Fig. 7 The σ_{23} distribution for the $[45/-45]_s$ laminate.

only discretization, implying a lower computational effort than other numerical methods where a very high number of degrees of freedom need to be considered to achieve realistic predictions of the interlaminar stress field. The present analysis appears to confirm the existence of stress singularities at the free edge that deserves a more detailed numerical investigation to determine the nature of such a singularity.[5,6,8,15,29]

Conclusions

This paper presents a general boundary integral formulation for the analysis of multilayered, generally stacked composite laminates under axial extension. The stress equilibrium equations, strain compatibility, traction boundary conditions, and ply interface continuity conditions are all exactly satisfied in the formulation and then numerically exact solutions are actually obtained. The boundary element method is employed to numerically solve the model and to calculate the interlaminar stress distribution. This approach makes it possible to analyze the laminate stress field under the considered load condition whatever the laminate composition and cross section. Moreover, the proposed method works in the context of no aprioristic assumptions, and it presents some computational advantages when compared with other numerical methods. A good agreement between the present results and those of other authors is observed. The stress fields show high gradients in the free edge regions, confirming the well-known singular trend at the interface free edge location. The proposed formulation solves the composite laminate elasticity problem, and it is able to accurately and efficiently describe the laminate response when it is subjected to axial extension. The formulation is consistent because the difference between the numerical solution, obtained by the boundary element method, and that of the continuous model only depends on the mesh refinement. The formulation is also able to be extended to more complex configurations and load conditions.

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